TS7: Thematic Session: Commutative Monoids

Thursday 3 July, 14:00–16:00 • Room 106

M. Ángeles Moreno Frías (Universidad de Cádiz, Spain)
Time: 14:00–14:20

Elasticity of a numerical semigroup and packed semigroups

Let S be a numerical semigroup and $\mathrm{msg}(S) = \{n_1, \cdots, n_e\}$ its minimal system of generators. Then $\mathrm{m}(S) = \min(\mathrm{msg}(S))$, $\mathrm{m}(S) = \max(\mathrm{msg}(S))$, and $\mathrm{e}(S)$ the cardinality of $(\mathrm{msg}(S))$, are called the multiplicity, comultiplicity and embedding dimension of S, respectively.

If $s \in S$ denote by $\mathsf{L}(s) = \{\lambda_1 + \cdots + \lambda_e \mid (\lambda_1, \cdots, \lambda_e) \in \mathbb{N}^e$ and $\lambda_1 n_1 + \cdots + \lambda_e n_e = s\}$. Define the *elasticity* of s as $\mathscr{S}(s) = \frac{\max(\mathsf{L}(s))}{\min(\mathsf{L}(s))}$. The *elasticity* of a numerical semigroup S, is defined as $\mathscr{S}(S) = \max\{\mathscr{S}(s) \mid s \in S\}$. In [2] we can see that the computation of the elasticity of a numerical semigroup is easy, because given a numerical semigroup S, in [2, Example 3.1.6] is shown that $\mathscr{S}(S) = \frac{\mathrm{m}(S)}{\mathrm{m}(S)}$.

In this talk, we study the following three sets:

- $\mathscr{L}(\mathrm{m}(S) = m, \mathrm{m}(S) = M) = \{S \mid S$ is a numerical semigroup, $\mathrm{m}(S) = m$ and $\mathrm{m}(S) = M\}$. In particular, we present an algorithm based in [3] to compute all its elements.
- $\mathscr{L}(\mathrm{m}(S) = m, \mathscr{S}(S) \leq q) = \{S \mid S$ is a numerical semigroup, $\mathrm{m}(S) = m$ and $\mathscr{S}(S) \leq q\}$.

  Following the notation introduced in [1], a *packed numerical semigroup* is a numerical semigroup S such that $\mathrm{m}(S) \leq 2\,\mathrm{m}(S) - 1$. We denote by $\mathscr{P}(\mathrm{m}(S) = m, \mathscr{S}(S) \leq q) = \{S \in \mathscr{L}(\mathrm{m}(S) = m, \mathscr{S}(S) \leq q) \mid S$ is a packed numerical semigroup$\}$. Over the set $\mathscr{L}(\mathrm{m}(S) = m, \mathscr{S}(S) \leq q)$, we define an equivalence binary relation $\mathscr{R}$ and we will see that

  $$\frac{\mathscr{L}(\mathrm{m}(S) = m, \mathscr{S}(S) \leq q)}{\mathscr{R}} = \{[S] \mid S \in \mathscr{P}(\mathrm{m}(S) = m, \mathscr{S}(S) \leq q)\}.$$

  Therefore, to compute all the elements of $\mathscr{L}(\mathrm{mul}{=}m, \mathrm{elas} \leq q)$, it is enough:

  (1) Compute $\mathscr{P}(\mathrm{mul}{=}m, \mathrm{elas} \leq q)$.
  (2) For every $\Delta \in \mathscr{P}(\mathrm{mul}{=}m, \mathrm{elas} \leq q)\}$, compute $[\Delta]$.

  We present some algorithmic procedures which allows to compute (1) and (2).

- $\mathscr{L}(\mathrm{e}(S) = 3, \mathscr{S}(S)) = q) = \{S \mid S$ is a numerical semigroup, $\mathrm{e}(S) = 3$ and $\mathscr{S}(S) = q\}$.
  We will see that if $q = \frac{b}{a}$ with $\{a, b\} \subseteq \mathbb{N}$ and $\gcd\{a, b\} = 1$, then the set

  $$\{\{\mathscr{L}(\mathrm{e}(S) = 3, \mathrm{m}(S) = ka, \mathrm{m}(S) = kb) \mid k \in \mathbb{N} \backslash \{0\}\}$$

  is a partition of $\mathscr{L}(\mathrm{e}(S) = 3, \mathscr{S}(S) = q)$. This fact allows us to describe some algorithmic procedures to compute $\mathscr{L}(\mathrm{e}(S) = 3, \mathscr{S}(S) = q)$.

[1] García-García, J.I.; Marín-Aragón, D.; Moreno-Frías, M.A. ; Rosales J.C. and Vigneron-Tenorio A., *Semigroups with fixed multiplicity and embedding dimension* Ars Math. Contemp. 17 (2019), 397–417. https://doi.org/10.26493/1855-3974.1937.5ea

[2] Geroldinger, A.; Halter-Koch F., *Non-Unique Factorizations Algebraic, Combinatorial and Analytic Theory,* Pure and Applied Mathematics 278 Chapman & Hall/CRC, 2006.

[3] Branco, M.B.; Ojeda I. and Rosales J.C., *The set of numerical semigroups of a given multiplicity and Frobenius number,* Port. Math. 78 (2001), 147–167. https://doi.org/10.4171/pm/2064

(*) Joint work with Prof. José Carlos Rosales (Universidad de Granada, Spain)
(**) This research is partially supported by Proyecto "Monoides y semigrupos afines (ProyExcel_00868)", Proyecto financiado en la convocatoria 2021 de Ayudas a Proyectos de Excelencia, en régimen de concurrencia competitiva, destinadas a entidades calificadas como Agentes del Sistema Andaluz del Conocimiento, en el ámbito del Plan Andaluz de Investigación, Desarrollo e Innovación (PAIDI 2020). Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía.

Isabel Colaço (Instituto Politécnico de Beja, Portugal)
Time: 14:20–14:40

Minimal free resolution of generalized repunit algebras

Let $\Bbbk$ be an arbitrary field and let $b > 1, n > 1$ and a be three positive integers. In this note we explicitly describe a minimal $S-$graded free resolution of the semigroup algebra $\Bbbk[S]$ when S is a generalized repunit numerical semigroup, that is, when S is the submonoid of $\mathbb{N}$ generated by $\{a_1, a_2, \dots, a_n\}$ where $a_1 = \sum_{j=0}^{n-1} b^j$ and $a_i - a_{i-1} = a\, b^{i-2},\ i = 2, \dots, n$, with $\gcd(a, a_1) = 1$.

(*) Joint work with Prof. Ignacio Ojeda (Universidad de Extremadura, Spain)

Alberto Vigneron-Tenorio (Universidad de Cádiz, Spain)
Time: 14:40–15:00

Generalisation of some results for numerical semigroups to affine semigroup

Strong numerical semigroups are introduced in [1]: a numerical semigroup S is strong if $x + y - m(S) \in S$ for all $x, y \in S$ such that $x \not\equiv y \mod m(S)$, where $m(S)$ is the multiplicity of S.

In this talk we generalize this concept to affine semigroups and focuses on some main similarities and differences between strong numerical semigroups and strong affine semigroups. Some results on strong affine semigroups are shown.

[1] Robles-Pérez, A. M.; Rosales, J. C., *Modular Frobenius pseudo-varieties*, Collect. Math. 74 (2023), 133–147. https://doi.org/10.1007/s13348-021-00339-0

(*) This ongoing work began during a research visit by Raquel Tapia-Ramos (Universidad de Cádiz) and Alberto Vigneron-Tenorio to University of Évora in February 2025.
(**) This research is partially supported by Proyecto "Monoides y semigrupos afines (ProyExcel_00868)", Proyecto financiado en la convocatoria 2021 de Ayudas a Proyectos de Excelencia, en régimen de concurrencia competitiva, destinadas a entidades calificadas como Agentes del Sistema Andaluz del Conocimiento, en el ámbito del Plan Andaluz de Investigación, Desarrollo e Innovación (PAIDI 2020). Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía.

Raquel Tapia-Ramos (Universidad de Cádiz, Spain)
Time: 15:00–15:20

An approach to ideals of affine semigroups and MED-semigroups

A subset P of an affine semigroup S is an ideal of S if $P + S \subseteq P$. This talk, based on the work presented in [1], focuses on the study of ideals of affine semigroups, providing characterizations and developing algorithms to compute all ideals that are also affine semigroups satisfying certain properties. Special emphasis is given to affine semigroups with maximal embedding dimension (MED-semigroups), which are defined by the unique property that all elements in the intersection of their Apéry sets, excluding zero, are minimal generators of S.

[1] García-García, J. I.; Tapia-Ramos, R.; Vigneron-Tenorio, A.: *On ideals of affine semigroups and affine semigroups with maximal embedding dimension.* Open Mathematics (2024) 22, no. 1: 20140101. https://doi.org/10.48550/arXiv.2405.14648

(*) Joint work with Prof. Alberto Vigneron-Tenorio (Universidad de Cádiz, Spain)
(**) This research is partially supported by Proyecto "Monoides y semigrupos afines (ProyExcel_00868)", Proyecto financiado en la convocatoria 2021 de Ayudas a Proyectos de Excelencia, en régimen de concurrencia competitiva, destinadas a entidades calificadas como Agentes del Sistema Andaluz del Conocimiento, en el ámbito del Plan Andaluz de Investigación, Desarrollo e Innovación (PAIDI 2020). Consejería de Universidad, Investigación e Innovación de la Junta de Andalucía.

Carlos Jesús Moreno (Universidad de Extremadura, Spain)
Time: 15:20–15:40

Numerical semigroups linked to curves with only one place at infinity

Let C be a curve on $\mathbb{P}^2$ with only one place at infinity at a point $p \in \mathbb{P}^2$, and let $S_{C,\infty}$ be its semigroup at infinity, i.e. the additive submonoid of $(\mathbb{N}, +)$ consisting of the orders —with negative sign— of the poles of the regular functions around (but not in) p. After a theorem by Abhyankar and Moh, we can associate to C a so-called δ-sequence in $\mathbb{N}_{>0}$ which is a system of generators of $S_{C,\infty}$, by no means unique.

Curves with only one place at infinity are relevant, for instance they play an important role in the study of the Jacobian conjecture. However, not so much is known about $S_{C,\infty}$ from a combinatorial point of view. In this talk we review the previous concepts and results and we see some properties of the δ-sequences; in particular, we introduce the notion of minimal δ-sequence as that generated only by the minimal elements of the semigroup at infinity. In addition, we show how we can compute the remaining δ-sequences associating with the same semigroup at infinity. Finally, we see algorithm procedures to compute minimal δ-sequences.

[1] Galindo, C.; Monserrat, F.; Moreno-Ávila, C.-J.; Moyano-Fernández, J.-J., Surfaces and semigroups at infinity. ArXiv:2408.15937. https://arxiv.org/pdf/2408.15931

(*) This talk is based on a joint work [1] with C. Galindo, F. Monserrat and J.-J. Moyano-Fernández.
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José Navarro (Universidad de Extremadura, Spain)
Time: 15:40–16:00

Exploring Toric Varieties with Condensed Mathematics: Some Open Problems

Affine toric varieties are central objects in algebraic geometry, with their structure deeply encoded by the combinatorics of their semigroup of characters. When we consider these varieties over general topological fields K (like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{C}_p$, or $\mathbb{F}_q((t))$), their K-points exhibit rich and often complicated analytic and topological structures that go beyond the purely algebraic picture.

Condensed mathematics, a new framework recently developed by Dustin Clausen and Peter Scholze, offers a powerful lens to study such topological-algebraic objects. By replacing topological spaces with 'condensed sets' (sheaves on a site of profinite sets), it provides categories with better formal properties, particularly suited for analytic geometry over topological fields.

This talk will provide a brief, non-technical introduction to the motivation and core ideas behind condensed mathematics, aimed at toric geometers. We will then briefly explore some open problems related to understanding affine toric varieties through this new perspective, such as the computation of condensed cohomology and its relation to known invariants.

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